



Stability estimate for an inverse boundary coefficient problem in thermal imaging

Mourad Bellassoued^a, Jin Cheng^b, Mourad Choulli^{c,*}

^a *Département de Mathématiques, Université des Sciences de Bizerte, 7021 Jarzouna Bizerte, Tunisia*

^b *School of Mathematical Sciences, Fudan University, Shanghai 200433, China*

^c *Laboratoire LMAM, UMR 7122, Université Paul Verlaine-Metz et CNRS, Ile du Saulcy, 57045 Metz cedex, France*

Received 14 November 2007

Available online 31 January 2008

Submitted by P. Sacks

Abstract

We establish a stability estimate for an inverse boundary coefficient problem in thermal imaging. The inverse problem under consideration consists in the determination of a boundary coefficient appearing in a boundary value problem for the heat equation with Robin boundary condition (we note here that the initial condition is assumed to be a priori unknown). Our stability estimate is of logarithmic type and it is essentially based on a logarithmic estimate for a Cauchy problem for the Laplace equation.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Inverse boundary coefficient problem; Robin boundary condition; Logarithmic stability estimate; Thermal imaging; Heat equation

1. Introduction

The thermal imaging is one of the most popular tools in non-destructive evaluation. In the present paper we are interested in one model coming from the mathematical formulation of the problem of detecting corrosion and damage in an inaccessible portion of some material object by thermal imaging. From the practical point of view, the thermal imaging approach of this problem is the following: a prescribed heat flux is applied to an accessible portion of the surface of the object and the resulting response is measured. From the information we have from the measurements we try to get properties and/or the shape of the damaged portion.

The object is represented by a bounded domain Ω of \mathbb{R}^n . We assume that $\Gamma = \partial\Omega$ consists in two disjoint parts Γ_i and Γ_e , each of them is with non-empty interior. The applied heat flux $g(x, t)$ has its support contained in $\tilde{\Gamma} \subset \Gamma_e$. The propagation of the heat through Ω is given by the following boundary value problem:

$$\begin{cases} \partial_t u(x, t) = \Delta u(x, t), & x \in \Omega, \quad t > 0, \\ \partial_\nu u(x, t) = g(x, t), & x \in \Gamma_e, \\ \partial_\nu u(x, t) + \alpha(x)u(x, t) = 0, & x \in \Gamma_i, \quad t > 0. \end{cases} \quad (1.1)$$

* Corresponding author.

E-mail address: choulli@univ-metz.fr (M. Choulli).

Here ∂_ν denotes the derivative in the direction of the exterior unit normal vector ν . As explained in [4] the Robin boundary condition $\partial_\nu u + \alpha u = 0$ corresponds to a Newton-cooling type of heat loss on the boundary with ambient temperature scaled to zero.

Let us note that we do not impose an initial condition in (1.1) because here and in the most industrial inverse problems the initial condition is usually unknown.

Let Γ_1 be a closed subset of Γ_e with non-empty interior. Our inverse problem can be formulated as follows: determine α and Γ_i from the boundary measurements

$$u(x, t) = h(x, t), \quad x \in \Gamma_1, \quad t > 0. \quad (1.2)$$

Other models involving mixed boundary conditions are possible. This is the case for instance in steel industries, where the problem consists in detecting corrosion inside a container from a thermal image outside. The container is represented by a bounded domain Ω of \mathbb{R}^n . Its outer boundary is denoted by Γ_e . The inside boundary of the container, where the corrosion may happen, is denoted by Γ_i . The temperature distribution, denoted by u , satisfies the following boundary value problem

$$\begin{cases} \partial_t u(x, t) = \Delta u(x, t), & x \in \Omega, \quad t > 0, \\ u(x, t) = c_0, & x \in \Gamma_i, \quad t > 0, \\ \partial_\nu u(x, t) = \beta(u^4 - u_a^4), & x \in \Gamma_e, \quad t > 0, \end{cases} \quad (1.3)$$

where c_0 , β , u^4 and u_a^4 are some given constants. The inverse problem is again the determination of the unknown inner boundary Γ_i from the overdetermined data (1.2).

Various uniqueness results of the determination of Γ_i in (1.1)–(1.2) were obtained in [3] and [4]. Uniqueness, stability and a numerical algorithm were proposed in [7] for the determination of Γ_i in (1.1)–(1.3) in the one-dimensional case. More results for the stationary case can be found in the literature, where different kinds of methods are employed to establish uniqueness, stability and numerical algorithms, we refer to [1,2,5,6,8,10] and references therein for more details.

Our aim in the present work is the stability issue for the inverse problem consisting in the determination of the heat loss coefficient in (1.1) from the boundary measurements (1.2). That is we assume that Γ_i is a priori known. The question of the stability in the determination of Γ_i from (1.2) will be discussed in another work.

In the rest of this paper we assume for simplicity that the boundary of Ω is of class C^∞ .

2. Stability estimate for an inverse boundary coefficient problem in an elliptic equation

We begin by recalling the following estimate

Theorem 2.1. (See [14].) Let $\beta \in]0, 1[$ and γ be a closed subset of Γ with non-empty interior. Then there exist positive constants $C_0 = C_0(\beta, \Omega, \gamma)$ and $C_1 = C_1(\Omega, \gamma)$ such that for all $v \in H^2(\Omega)$,

$$\|v\|_{H^1(\Omega)} \leq C_0 \frac{\|v\|_{H^2(\Omega)}}{[\ln(C_1 \frac{\|v\|_{H^2(\Omega)}}{\|\Delta v\|_{L^2(\Omega)} + \|v\|_{L^2(\gamma)} + \|\partial_\nu v\|_{L^2(\gamma)})}]^\beta}. \quad (2.1)$$

From this theorem we derive the following estimate for the Cauchy problem for the Laplace operator.

Corollary 2.2. Let $\beta \in]0, 1[$, $M > 0$, and Γ_0, Γ_1 be two closed subsets of Γ with non-empty interior. Then there exist positive constants $A = \tilde{A}(\beta, \Omega, \Gamma_0, \Gamma_1)M$ and $B = \tilde{B}(\Omega, \Gamma_0)M$ such that for all $v \in H^3(\Omega)$ satisfying $\Delta v = 0$ and $\|v\|_{H^3(\Omega)} \leq M$, we have

$$\|v\|_{L^2(\Gamma_1)} + \|\partial_\nu v\|_{L^2(\Gamma_1)} \leq \frac{A}{[\ln(\frac{B}{\|v\|_{L^2(\Gamma_0)} + \|\partial_\nu v\|_{L^2(\Gamma_0)}})]^\beta}.$$

Proof. This is immediate from estimate (2.1), the following interpolation inequality (e.g. [12])

$$\|w\|_{H^2(\Omega)} \leq \|w\|_{H^1(\Omega)}^{1/2} \|w\|_{H^3(\Omega)}^{1/2}, \quad \text{if } w \in H^3(\Omega),$$

and the continuity of the trace operator

$$w \in H^2(\Omega) \rightarrow (w|_{\Gamma_1}, \partial_\nu w|_{\Gamma_1}) \in L^2(\Gamma_1) \times L^2(\Gamma_1). \quad \square$$

We can now use this corollary to establish a logarithmic stability estimate for an inverse elliptic boundary coefficient problem.

$$\begin{cases} \Delta v(x) = 0, & x \in \Omega, \\ \partial_\nu v(x) = g(x), & x \in \Gamma_e, \\ \partial_\nu v(x) + \alpha(x)v(x) = 0, & x \in \Gamma_i. \end{cases} \quad (2.2)$$

This model describe for instance the electrostatics of a conductor Ω having an inaccessible part of his boundary, denoted by Γ_i , affected by corrosion. Here v represents the electrostatic potential, g the prescribed current density on the accessible part of the boundary Γ_e ; while α , called the coefficient of corrosion, represents the characteristic of corrosion damage.

The inverse problem consists in the determination of the boundary coefficient α from the boundary measurements

$$v(x) = h(x) \quad \text{on } \Gamma_0,$$

where Γ_0 is a subset of Γ_e .

Before we state the stability estimate for the above inverse problem, we need to introduce the space to which belongs the unknown coefficient α . Let m be a positive integer, $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. We consider the vector space

$$B_{s,p}(\mathbb{R}^m) = \{w \in \mathcal{S}'(\mathbb{R}^m); (1 + |\xi|^2)^{s/2} \mathcal{F}w \in L^p(\mathbb{R}^m)\},$$

where $\mathcal{S}'(\mathbb{R}^m)$ is the space of temperate distributions on \mathbb{R}^m and $\mathcal{F}w$ is the Fourier transform of w . Equipped with the norm

$$\|w\|_{B_{s,p}(\mathbb{R}^m)} = \|(1 + |\xi|^2)^{s/2} \mathcal{F}w\|_{L^p(\mathbb{R}^m)},$$

$B_{s,p}(\mathbb{R}^m)$ is a Banach space (it is noted that $B_{s,2}(\mathbb{R}^m)$ is merely the Sobolev space $H^s(\mathbb{R}^m)$). Using local chart and partition of unity, we construct $B_{s,p}(\Gamma)$ from $B_{s,p}(\mathbb{R}^{n-1})$ in the same way as $H^s(\Gamma)$ is built from $H^s(\mathbb{R}^{n-1})$.

Next we recall a regularity result of the weak solution to the boundary value problem (2.2). To this end we consider the assumptions, where $k \geq 0$ is an integer,

- (A) $\tilde{\alpha} = \chi_{\Gamma_i} \alpha \in B_{k+1/2,1}(\Gamma) \cap L^\infty(\Gamma)$ is positive and there exist an open subset γ of Γ and a positive constant κ such that $\alpha \geq \kappa$ a.e. on γ .
- (B) $\tilde{g} = \chi_{\Gamma_e} g \in H^{k+1/2}(\Gamma)$.

Theorem 2.3. (See [9].) *We assume that α and g satisfy the assumptions (A) and (B). Then the boundary value problem (2.2) has a unique solution $v = v_{\alpha,g} \in H^{2+k}(\Omega)$. Moreover, if*

$$M \geq \|\tilde{\alpha}\|_{B_{k+1/2,1}(\Gamma)},$$

then v satisfies the estimate

$$\|v\|_{H^{k+2}(\Omega)} \leq C \|\tilde{g}\|_{H^{k+1/2}(\Gamma)}.$$

Here C is a constant depending only on M , Ω , γ and κ , where γ and κ are as in (A).

Let $\alpha = \alpha_i$, $i = 1, 2$, and g satisfy (A) and (B) with $k \geq 1$, where γ and κ are the same for α_1 and α_2 . We assume in addition that there exists a positive constant M such that

$$M \geq \|\tilde{\alpha}_1\|_{B_{3/2,1}(\Gamma)} + \|\tilde{\alpha}_1\|_{L^\infty(\Gamma)}, \quad \|\tilde{\alpha}_2\|_{B_{3/2,1}(\Gamma)} + \|\tilde{\alpha}_2\|_{L^\infty(\Gamma)}. \quad (2.3)$$

Note that we used $B_{k+1/2,1}(\Gamma) \subset B_{3/2,1}(\Gamma)$.

Let $v_i = v_{\alpha_i,g} \in H^{k+2}(\Omega) \subset H^3(\Omega)$. It follows from Theorem 2.2 (with $k = 1$) that

$$\|v_i\|_{H^3(\Omega)} \leq C,$$

where C is a constant depending only on Ω , γ , κ and M .

Next we choose k such that $k + 2 > n/2$. Under this condition, it follows that $H^{k+2}(\Omega)$ is continuously embedded in $C(\bar{\Omega})$ (e.g. [12]). We take then K a compact subset of $\{x \in \Gamma_i; v_1(x) \neq 0\}$. Starting from the following elementary identity

$$v_1(\alpha_1 - \alpha_2) = (v_2 - v_1)\alpha_2 + (\partial_v v_2 - \partial_v v_1) \quad \text{on } \Gamma_i,$$

we get

$$\begin{aligned} \|\alpha_1 - \alpha_2\|_{L^2(K)} &\leq \left\| \frac{1}{v_1} \right\|_{L^\infty(K)} [\|v_2 - v_1\|_{L^2(K)} \|\tilde{\alpha}_2\|_{L^\infty(\Gamma)} + \|\partial_v v_2 - \partial_v v_1\|_{L^2(K)}] \\ &\leq \left\| \frac{1}{v_1} \right\|_{L^\infty(K)} \max(M, 1) [\|v_2 - v_1\|_{L^2(K)} + \|\partial_v v_2 - \partial_v v_1\|_{L^2(K)}]. \end{aligned}$$

In view of Corollary 2.1 and the last inequality we can state the following result.

Theorem 2.4. Let $\alpha = \alpha_i$, $i = 1, 2$, and g satisfy (A) and (B) with $k \geq 1$ is such that $k + 2 > n/2$, where γ and κ are the same for α_1 and α_2 . Let the assumption (2.3) be satisfied. Let K be a compact subset of $\{x \in \Gamma_i; v_1(x) \neq 0\}$. Let $\beta \in]0, 1[$. Then there exist two constants $C_0 = \tilde{C}_0(\Omega, \gamma, \kappa, \beta, \Gamma_i, \Gamma_0, K, v_1)M$ and $C_1 = \tilde{C}_1(\Omega, \gamma, \kappa, \Gamma_0)M$ such that

$$\|\alpha_1 - \alpha_2\|_{L^2(K)} \leq \frac{C_0}{[\ln(\frac{C_1}{\|v_1 - v_2\|_{L^2(\Gamma_0)}})]^\beta}.$$

In two-dimensional case we have a better estimate. Precisely, we proved in [6] the following result. We consider the assumptions, where $\theta \in]0, 1[$,

(A') $\tilde{\alpha} = \chi_{\Gamma_i} \alpha \in C^{1,\theta}(\Gamma)$, $\alpha \geq 0$, and non-identically equal to zero,

(B') $\tilde{g} = \chi_{\Gamma_e} g \in C^{1,\theta}(\Gamma)$.

Theorem 2.5. Let $M > 0$ be a given constant. For $i = 1, 2$, let α_i satisfy (A') and $\|q_i\|_{C^{1,\theta}(\Gamma)} \leq M$. Assume that g satisfies (B') and it is non-identically equal to zero. Let K be a compact subset of $\{x \in \Gamma_i; v_1(x) \neq 0\}$ and Γ_0 be an open subset of Γ_e with non-empty interior. If $\|\alpha_1 - \alpha_2\|_{C^{1,\alpha}(\Gamma)}$ is sufficiently small, then there exist positive constants $C'_0 = \tilde{C}'_0(\Omega, \gamma, \kappa, \beta, \Gamma_i, \Gamma_0, K, v_1)M$ and $C'_1 = \tilde{C}'_1(\Omega, \gamma, \kappa, \Gamma_0)M$ such that

$$\|\alpha_1 - \alpha_2\|_{L^2(K)} \leq \frac{C'_0}{\ln(\frac{C'_1}{\|v_1 - v_2\|_{L^2(\Gamma_0)}})}.$$

3. Stability estimate for an inverse boundary coefficient problem in a parabolic equation

We use the result of the preceding section to establish a stability estimate for the inverse problem in thermal imaging described in the introduction. We need to assume that the boundary function g in (1.1) depends only on the space variable. This is a reasonable assumption. Because this means that we impose at every time the same heat flux on a part of the accessible boundary Γ_e . The case of the time dependent heat flux will be discussed in the next section.

If α satisfy assumption (B) (see the previous section) then the bilinear form

$$a_\alpha(u, v) = \int_{\Omega} Du \cdot Dv \, dx + \int_{\Gamma_i} \alpha uv \, d\sigma, \quad u, v \in H^1(\Omega),$$

is bounded and coercive. Therefore the spectrum of the operator A_α corresponding to the Laplace operator with Robin boundary condition $\partial_\nu u + \alpha \chi_{\Gamma_i} u = 0$ on Γ consists in a sequence of eigenvalues

$$0 < \lambda_\alpha^1 \leq \lambda_\alpha^2 \leq \dots \leq \lambda_\alpha^l \rightarrow +\infty.$$

If B is the Laplace operator with Robin boundary condition $\partial_\nu u + \kappa \chi_\gamma u = 0$, where κ and γ are the same as in assumption (A) then, similarly as for A_α , the spectrum of B consists in a sequence of eigenvalues

$$0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_l \rightarrow +\infty.$$

In addition, according to the min–max theorem, we have $\mu_l \leq \lambda_\alpha^l$ for each $l \geq 1$ because $b(u, v) \leq a(u, v)$, where b is the bilinear form associated to B :

$$b(u, v) = \int_{\Omega} Du \cdot Dv \, dx + \int_{\Gamma_i} \kappa \chi_\gamma uv \, d\sigma, \quad u, v \in H^1(\Omega).$$

In particular, we have the following lower bound

$$\mu \leq \lambda_\alpha^l, \quad \text{for each } l \geq 1. \quad (3.1)$$

Here $\mu = \mu(\kappa, \gamma)$ is a constant (clearly we can take $\mu = \mu_1$).

Let (φ_α^l) be an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of A_α . We recall that $-A_\alpha$ generates an analytic semi-group e^{-tA_α} in the half space $\{z \in \mathbb{C}; \Re z > 0\}$. This semi-group is explicitly given by

$$e^{-tA_\alpha} f = \sum_{l \geq 1} e^{-t\lambda_\alpha^l} (\varphi_\alpha^l, f) \varphi_\alpha^l, \quad f \in L^2(\Omega),$$

where (\cdot, \cdot) is the usual scalar product on $L^2(\Omega)$.

Let α, g satisfy (A) and (B). We denote by $u_{\alpha, g}$ a solution of the boundary value problem (1.1) belonging to $C([0, +\infty[; L^2(\Omega)) \cap C([0, +\infty[; H^2(\Omega)) \cap L^\infty(0, +\infty; H^1(\Omega))$ and having the property

$$u_{\alpha, g}(\cdot, 0) \in L^2(\Omega), \quad \|u_{\alpha, g}(\cdot, 0)\|_{L^2(\Omega)} \leq R_0.$$

Here and in the sequel R_0 is a fixed non-negative constant.

We decompose $u_{\alpha, g}$ into two terms $u_{\alpha, g} = v_{\alpha, g} + w_{\alpha, g}$, where $w_{\alpha, g}$ is the solution of the following initial-boundary value problem

$$\begin{cases} \partial_t w(x, t) = \Delta w(x, t), & x \in \Omega, \, t > 0, \\ \partial_\nu w = 0, & x \in \Gamma_e, \\ \partial_\nu w(x, t) + \alpha(x)w(x, t) = 0, & x \in \Gamma_i, \, t > 0, \\ w(x, 0) = u_{\alpha, g}(x, 0) - v_{\alpha, g}, & x \in \Omega. \end{cases}$$

In the other words, $v_{\alpha, g}$ is the steady state solution to the problem when defining $u_{\alpha, g} = v_{\alpha, g} + w_{\alpha, g}$.

From the classical theory of analytic semi-groups (e.g. [13] or [15]) we know that there exists a non-negative constant C , not depending on α , such that

$$\|(-A_\alpha)^{1/2} e^{-tA_\alpha}\| \leq C \frac{e^{-\mu t}}{\sqrt{t}}, \quad t > 0,$$

where $\|\cdot\|$ is the operator norm. On the other hand following [11] we have $D((-A)^{1/2}) = H^1(\Omega)$. Therefore

$$\|e^{-tA_\alpha} h\|_{H^1(\Omega)} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|h\|_{L^2(\Omega)}, \quad t > 0, \, h \in L^2(\Omega). \quad (3.2)$$

We recall that from the estimate in Theorem 2.2 we have

$$\|v_{\alpha, g}\|_{H^3(\Omega)} \leq R_1,$$

where $R_1 = \tilde{R}_1(M, \Omega, \kappa, \gamma) \|\tilde{g}\|_{H^{3/2}(\Gamma)}$ and M is such that $M \geq \|\tilde{\alpha}\|_{B_{3/2,1}(\Gamma)}$.

In view of (3.2) we find

$$\|w_{\alpha, g}(\cdot, t)\|_{H^1(\Omega)} = \|e^{-tA_\alpha} w_{\alpha, g}(\cdot, 0)\|_{H^1(\Omega)} \leq C(R_0 + R_1) \frac{e^{-\mu t}}{\sqrt{t}} = \tilde{C} \frac{e^{-\mu t}}{\sqrt{t}}. \quad (3.3)$$

Now for $i = 1, 2$, let α_i and g satisfy (A) and (B). We assume in addition that the assumption (2.3) is satisfied. We introduce the following notations $u_i = u_{\alpha_i, g}$ and $v_i = v_{\alpha_i, g}$, $i = 1, 2$. Let A be the norm of the trace operator

$$w \in H^1(\Omega) \rightarrow w|_{\Gamma_0} \in L^2(\Gamma_0).$$

From (3.3) it follows, where $t > 0$,

$$\begin{aligned}\|v_1 - v_2\|_{L^2(\Gamma_0)} &\leq \|v_1 - u_1(\cdot, t)\|_{L^2(\Gamma_0)} + \|v_2 - u_2(\cdot, t)\|_{L^2(\Gamma_0)} + \|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^2(\Gamma_0)} \\ &\leq \Lambda \|v_1 - u_1(\cdot, t)\|_{H^1(\Omega)} + \Lambda \|v_2 - u_2(\cdot, t)\|_{H^1(\Omega)} + \|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^2(\Gamma_0)} \\ &\leq 2\Lambda \tilde{C} \frac{e^{-\mu t}}{\sqrt{t}} + \|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^2(\Gamma_0)} \\ &\leq 2\Lambda \tilde{C} \frac{e^{-\mu t}}{\sqrt{t}} + \|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))}.\end{aligned}$$

Passing to the limit when t tends to infinity, we find

$$\|v_1 - v_2\|_{L^2(\Gamma_0)} \leq \|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))}.$$

This estimate and Theorem 2.2 lead to

Theorem 3.1. Let $\alpha = \alpha_i$, $i = 1, 2$, and g satisfy (A) and (B) with $k \geq 1$ is such that $k + 2 > n/2$, where γ and κ are the same for α_1 and α_2 . Let the assumption (2.3) be satisfied. Let K be a compact subset of $\{x \in \Gamma_i; v_1(x) \neq 0\}$. Let $\beta \in]0, 1[$. Then there exist two constants $C_0 = \tilde{C}_0(\Omega, \gamma, \kappa, \beta, \Gamma_i, \Gamma_0, K, v_1)M$ and $C_1 = \tilde{C}_1(\Omega, \gamma, \kappa, \Gamma_0)M$ such that

$$\|\alpha_1 - \alpha_2\|_{L^2(K)} \leq \frac{C_0}{[\ln(\frac{C_1}{\|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))}})]^\beta}.$$

In the two-dimensional case we have also the following result.

Theorem 3.2. Let $M > 0$ be a given constant. For $i = 1, 2$, let α_i satisfies (A') and $\|\alpha_i\|_{C^{1,\theta}(\Gamma)} \leq M$. Assume that g satisfies (B') and it is non-identically equal to zero. Let K be a compact subset of $\{x \in \Gamma_i; v_1(x) \neq 0\}$ and Γ_0 be an open subset of Γ_e with non-empty interior. If $\|\alpha_1 - \alpha_2\|_{C^{1,\alpha}(\Gamma)}$ is sufficiently small, then there exist positive constants $C'_0 = \tilde{C}'_0(\Omega, \gamma, \kappa, \beta, \Gamma_i, \Gamma_0, K, v_1)M$ and $C'_1 = \tilde{C}'_1(\Omega, \gamma, \kappa, \Gamma_0)M$ such that

$$\|\alpha_1 - \alpha_2\|_{L^2(K)} \leq \frac{C'_0}{\ln(\frac{C'_1}{\|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))}})}.$$

4. The case of time dependent heat flux

We will use the same notations as in the preceding section.

We first make the following assumption:

(B'') $\tilde{h} = \chi_{\Gamma_1} h \in C([0, +\infty[; H^{3/2}(\Gamma))$ and $\partial_t \tilde{g} \in C([0, +\infty[; H^{1/2}(\Gamma))$.

Let α and h satisfy (A) and (B''). For each $t \geq 0$, we easily derive from Theorem 2.2 that the boundary value problem

$$\begin{cases} \Delta y(x, t) = 0, & x \in \Omega, \\ \partial_\nu y(x, t) = h(x, t), & x \in \Gamma_e, \\ \partial_\nu y(x, t) + \alpha(x)y(x, t) = 0, & x \in \Gamma_i, \end{cases}$$

has a unique solution $y_{\alpha, h} \in C([0, +\infty[; H^3(\Omega)))$. Moreover $\partial_t y_{\alpha, h} \in C([0, +\infty[; H^2(\Omega)))$, $\partial_t y_{\alpha, h}$ is the solution of the boundary value problem

$$\begin{cases} \Delta w(x, t) = 0, & x \in \Omega, \\ \partial_\nu w(x, t) = \partial_t h(x, t), & x \in \Gamma_e, \\ \partial_\nu w(x, t) + \alpha(x)w(x, t) = 0, & x \in \Gamma_i, \end{cases}$$

and the following estimates hold

$$\|y_{\alpha,h}(\cdot, t)\|_{H^2(\Omega)} \leq C \|\tilde{h}(\cdot, t)\|_{H^{1/2}(\Gamma)}, \quad \|\partial_t y_{\alpha,h}(\cdot, t)\|_{H^2(\Omega)} \leq C \|\partial_t \tilde{h}(\cdot, t)\|_{H^{1/2}(\Gamma)}, \quad (4.1)$$

where the constant C depends only on M , $M \geq \|\tilde{\alpha}\|_{B_{3/2,1}(\Gamma)}$, γ , κ , as appearing in assumption (A).

We consider now the following initial-boundary value problem

$$\begin{cases} \partial_t u^0(x, t) = \Delta u^0(x, t), & x \in \Omega, \quad t > 0, \\ \partial_\nu u^0(x, t) = h(x, t), & x \in \Gamma_e, \\ \partial_\nu u^0(x, t) + \alpha(x)u^0(x, t) = 0, & x \in \Gamma_i, \quad t > 0, \\ u^0(x, 0) = 0, & x \in \Omega. \end{cases} \quad (4.2)$$

In a classical way we decompose the solution of the initial-boundary value problem (4.2), denoted by $u_{\alpha,h}^0$, in the following form

$$u_{\alpha,h}^0(x, t) = y_{\alpha,h}(x, t) + \sum_{l \geq 1} C_{\alpha,h}^l(t) \varphi_\alpha^l.$$

A simple calculation shows

$$C_{\alpha,h}^l(t) = -(y_{\alpha,h}(\cdot, 0), \varphi_\alpha^l) e^{-\lambda_\alpha^l t} - \int_0^t e^{-\lambda_\alpha^l(t-s)} (\partial_t y_{\alpha,h}(\cdot, s), \varphi_\alpha^l) ds.$$

Using the elementary convexity inequality $(a+b)^2 \leq 2(a^2 + b^2)$, we find

$$C_{\alpha,h}^l(t)^2 \leq 2e^{-2\mu t} (y_{\alpha,h}(\cdot, 0), \varphi_\alpha^l)^2 + 2 \int_0^t e^{-2\mu(t-s)} (y_{\alpha,h}(\cdot, s), \varphi_\alpha^l)^2 ds.$$

Therefore, for $k = 1, 2$,

$$\sum_{l \geq 1} (\lambda_\alpha^l)^k C_{\alpha,h}^l(t)^2 \leq 2e^{-2\mu t} \sum_{l \geq 1} (\lambda_\alpha^l)^k (y_{\alpha,h}(\cdot, 0), \varphi_\alpha^l)^2 + 2 \int_0^t e^{-2\mu(t-s)} \left[\sum_{l \geq 1} (\lambda_\alpha^l)^k (y_{\alpha,h}(\cdot, s), \varphi_\alpha^l)^2 \right] ds.$$

Or

$$w \rightarrow \left[\sum_{l \geq 1} (\lambda_\alpha^l)^k (w, \varphi_\alpha^l)^2 \right]^{1/2} \quad (4.3)$$

define an equivalent norm on $H^k(\Omega)$.

Then with respect to the norm defined by (4.3), we have

$$\begin{aligned} \|u_{\alpha,h}^0(\cdot, t)\|_{H^1(\Omega)} &\leq \|y_{\alpha,h}(\cdot, t)\|_{H^1(\Omega)} + \sqrt{2} e^{-\mu t} \|y_{\alpha,h}(\cdot, 0)\|_{H^1(\Omega)} \\ &\quad + \sqrt{2} \left\{ \int_0^t e^{-2\mu(t-s)} \|\partial_t y_{\alpha,h}(\cdot, s)\|_{H^1(\Omega)}^2 ds \right\}^{1/2}. \end{aligned}$$

In view of (4.1) this estimate implies

$$\|u_{\alpha,h}^0(\cdot, t)\|_{H^1(\Omega)} \leq C \left[\|\tilde{h}(\cdot, t)\|_{H^{1/2}(\Gamma)} + e^{-\mu t} \|\tilde{h}(\cdot, 0)\|_{H^{1/2}(\Gamma)} + \left\{ \int_0^t e^{-2\mu(t-s)} \|\partial_t \tilde{h}(\cdot, s)\|_{H^{1/2}(\Gamma)}^2 ds \right\}^{1/2} \right]. \quad (4.4)$$

Next let g satisfy assumption (B). Recall that the solution of the boundary value problem (2.2) is denoted by $v_{\alpha,g}$. Let $u_{\alpha,h}$ denote a solution of the boundary value problem (1.1) (with g replaced by h) belonging to $C([0, +\infty[; L^2(\Omega)) \cap C([0, +\infty[; H^2(\Omega)) \cap L^\infty(0, +\infty; H^1(\Omega))$ and satisfying

$$u_{\alpha,h}(\cdot, 0) \in L^2(\Omega), \quad \|u_{\alpha,h}(\cdot, 0)\|_{L^2(\Omega)} \leq R_0,$$

where R_0 is a fixed positive constant.

As in the previous section we write $u_{\alpha,h} = v_{\alpha,g} + w_{\alpha,h,g}$, where $w_{\alpha,h,g}$ is the solution of the boundary value problem

$$\begin{cases} \partial_t w(x, t) = \Delta w(x, t), & x \in \Omega, \ t > 0, \\ \partial_\nu w(x, t) = h(x, t) - g(x), & x \in \Gamma_e, \\ \partial_\nu w(x, t) + \alpha(x)w(x, t) = 0, & x \in \Gamma_i, \ t > 0, \\ w(x, 0) = u_{\alpha,h}(x, 0) - v_{\alpha,g}, & x \in \Omega. \end{cases}$$

We split $w_{\alpha,h,g}$ into two terms: $w_{\alpha,h,g} = u_{\alpha,h-g}^0 + \tilde{w}_{\alpha,h,g}$. Clearly $\tilde{w}_{\alpha,h,g}$ is the solution of the following boundary value problem

$$\begin{cases} \partial_t \tilde{w}(x, t) = \Delta \tilde{w}(x, t), & x \in \Omega, \ t > 0, \\ \partial_\nu \tilde{w}(x, t) = 0, & x \in \Gamma_e, \\ \partial_\nu \tilde{w}(x, t) + \alpha(x)\tilde{w}(x, t) = 0, & x \in \Gamma_i, \ t > 0, \\ \tilde{w}(x, 0) = u_{\alpha,h}(x, 0) - v_{\alpha,g}, & x \in \Omega. \end{cases}$$

Similarly to (3.3) we prove

$$\|\tilde{w}_{\alpha,h,g}\|_{H^1(\Omega)} \leq C_0 \frac{e^{-\mu t}}{\sqrt{t}}, \quad (4.5)$$

where $C_0 = \tilde{C}_0(\kappa, \gamma, R_0)M$ is a constant with $M \geq \|\alpha\|_{H^{3/2}(\Gamma)}$.

Combining the estimates (4.4) and (4.5) we find

$$\begin{aligned} \|w_{\alpha,h,g}(\cdot, t)\|_{H^1(\Omega)} &\leq C_0 \frac{e^{-\mu t}}{\sqrt{t}} + C \left[\|\tilde{h}(\cdot, t) - \tilde{g}\|_{H^{1/2}(\Gamma)} + e^{-\mu t} \|\tilde{h}(\cdot, 0)\|_{H^{1/2}(\Gamma)} \right. \\ &\quad \left. + \left\{ \int_0^t e^{-2\mu(t-s)} \|\partial_t \tilde{h}(\cdot, s)\|_{H^{1/2}(\Gamma)}^2 ds \right\}^{1/2} \right]. \end{aligned}$$

As we have done in the previous section, we can deduce from the last estimate the following result.

Theorem 4.1. Let $\alpha = \alpha_i$, $i = 1, 2$, g and h satisfy respectively (A), (B) and (B'') with $k \geq 1$ is such that $k + 2 > n/2$, where γ and κ are the same for α_1 and α_2 . Let the assumption (2.3) be satisfied and

$$\limsup_{t \rightarrow +\infty} \left[\|\tilde{h}(\cdot, t) - \tilde{g}\|_{H^{1/2}(\Gamma)} + \left\{ \int_0^t e^{-2\mu(t-s)} \|\partial_t \tilde{h}(\cdot, s)\|_{H^{1/2}(\Gamma)}^2 ds \right\}^{1/2} \right] = 0. \quad (4.6)$$

Let K be a compact subset of $\{x \in \Gamma_i; v_{\alpha_1,g}(x) \neq 0\}$. Let $\beta \in]0, 1[$. Then there exist two constants $C_0 = \tilde{C}_0(\Omega, \gamma, \kappa, \beta, \Gamma_i, \Gamma_0)M$ and $C_1 = \tilde{C}_1(\Omega, \gamma, \kappa, \Gamma_0)M$ such that

$$\|\alpha_1 - \alpha_2\|_{L^2(K)} \leq \frac{C_0}{[\ln(\frac{C_1}{\|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))}})]^\beta}.$$

Let l satisfy (B'') and there exists $\theta > 0$ such that

$$\sup_{t \geq 0} e^{t\theta} (\|\tilde{l}(\cdot, t)\|_{H^{3/2}(\Gamma)} + \|\partial_t \tilde{l}(\cdot, t)\|_{H^{1/2}(\Gamma)}) < \infty, \quad (4.7)$$

then $h = g + l$ satisfies (4.6).

We note that a particular case of a function satisfying (4.7) is given by $l(x, t) = \omega(t)\rho(x)$, where $\rho \in H^{3/2}(\Gamma)$ and

$$\lim_{t \rightarrow +\infty} e^{t\theta} \omega(t) = \lim_{t \rightarrow +\infty} e^{t\theta} \omega'(t) = 0.$$

References

- [1] G. Alessandrini, L. Del Piero, L. Rondi, Stable determination of corrosion by a single electrostatic measurement, *Inverse Problems* 19 (2003) 973–984.
- [2] G. Alessandrini, E. Sincich, Solving elliptic Cauchy problems and the identification of nonlinear corrosion, *J. Comput. Appl. Math.* 198 (2) (2007) 307–320.
- [3] K. Bryan, L.F. Caudill Jr., An inverse problem in thermal imaging, *SIAM J. Appl. Math.* 56 (1996) 715–735.
- [4] K. Bryan, L.F. Caudill Jr., Uniqueness for a boundary identification problem in thermal imaging, in: J. Graef, R. Shivaji, B. Soni, Zhu (Eds.), *Differential Equations and Computational Simulations III*, in: *Electron. J. Differ. Equ. Conf.*, vol. 01, 1997, pp. 23–39.
- [5] S. Chaabane, I. Fellah, M. Jaoua, J. Leblond, Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems, *Inverse Problems* 20 (2004) 47–59.
- [6] J. Cheng, M. Choulli, J. Lin, Stable determination of a boundary coefficient in an elliptic equation, *Math. Models Methods Appl. Sci.* 18 (1) (2008) 107–123.
- [7] J. Cheng, M. Choulli, Y. Wang, M. Yamamoto, An industrial inverse problem for the heat equation, in preparation.
- [8] J. Cheng, M. Choulli, X. Yang, An iterative BEM for the inverse problem of detecting corrosion in a pipe, *Numer. Math. J. Chinese Univ.* 14 (3) (2005) 252–266.
- [9] M. Choulli, Stability estimates for an inverse elliptic problem, *J. Inverse Ill-Posed Probl.* 10 (6) (2002) 601–610.
- [10] M. Choulli, An inverse problem in corrosion detection: Stability estimates, *J. Inverse Ill-Posed Probl.* 12 (4) (2004) 349–367.
- [11] D. Fujiwara, Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order, *Proc. Japan Acad.* 43 (1967) 82–86.
- [12] J.-L. Lions, E. Magenes, *Problèmes aux limites non-homogènes et applications*, vol. I, Dunod, Paris, 1968.
- [13] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, second ed., Springer-Verlag, Berlin, 1976.
- [14] K.-D. Phung, Remarques sur l’observabilité pour l’équation de Laplace, *ESAIM Control Optim. Calc. Var.* 9 (2003) 621–635.
- [15] M. Renardy, R.C. Rogers, *An Introduction to Partial Differential Equations*, Springer-Verlag, New York, 1993.